矩阵乘法加速

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设矩阵 $\mathbf{A} = (a_{ij})$ 和 $\mathbf{B} = (b_{ij})$ 是 n 阶方阵，乘积 $\mathbf{C} = (c_{ij})$ 亦是 n 阶方阵，其中

$$c_{ij} = \sum_{k \in [n]} a_{ik} b_{kj}$$

因此按标准的矩阵乘法，计算 $\mathbf{C}$ 的时间开销为 $\Omega(n^3)$，事实上这个时间复杂度是可以改进的。

1 分治递归

设计算 $\mathbf{C}$ 的时间开销为 $T(n)$，将矩阵分成 $2 \times 2 = 4$ 块，由分块矩阵乘法有

$$\begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11}\mathbf{B}_{11} + \mathbf{A}_{12}\mathbf{B}_{21} & \mathbf{A}_{11}\mathbf{B}_{12} + \mathbf{A}_{12}\mathbf{B}_{22} \\ \mathbf{A}_{21}\mathbf{B}_{11} + \mathbf{A}_{22}\mathbf{B}_{21} & \mathbf{A}_{21}\mathbf{B}_{12} + \mathbf{A}_{22}\mathbf{B}_{22} \end{bmatrix}$$

其中包含 8 个 $n/2$ 阶方阵相乘、4 个 $n/2$ 阶方阵相加，注意每个 $n/2$ 阶方阵有 $n^2/4$ 个元素，因此共需进行 n^2 次加法，综上有递推关系

$$T(n) = 8 \cdot T(n/2) + c_1 n^2$$

其中 c_1 为单次加法的时间开销。设 $n = 2^k$，则

$$\begin{aligned} T(2^k) &= 8^1 \cdot T(2^{k-1}) + 8^0 \cdot c_1 4^k \\ 8^1 \cdot T(2^{k-1}) &= 8^2 \cdot T(2^{k-2}) + 8^1 \cdot c_1 4^{k-1} \\ 8^2 \cdot T(2^{k-2}) &= 8^3 \cdot T(2^{k-3}) + 8^2 \cdot c_1 4^{k-2} \\ &\vdots \\ 8^{k-1} \cdot T(2^1) &= 8^k \cdot T(2^0) + 8^{k-1} \cdot c_1 4^1 \end{aligned}$$

注意 $8^k = n^3$，$T(1) = c_2$ 是单次乘法的时间开销，累加可得

$$\begin{aligned} T(n) &= c_2 n^3 + c_1 4^k + 2^1 \cdot c_1 4^k + 2^2 \cdot c_1 4^k + \cdots + 2^{k-1} \cdot c_1 4^k = c_2 n^3 + c_1 4^k \frac{1 - 2^k}{1 - 2} \\ &= c_2 n^3 + c_1 n^2 (n - 1) = (c_2 + c_1) n^3 - c_1 n^2 \end{aligned}$$

即直接分治递归并不能改进时间复杂度。

2 基本想法

要想改进时间复杂度，必须得减少子问题的个数，即乘法的次数。将乘积拉直，易知

$$\begin{bmatrix} \mathbf{A}_{11}\mathbf{B}_{11}+\mathbf{A}_{12}\mathbf{B}_{21} \\ \mathbf{A}_{11}\mathbf{B}_{12}+\mathbf{A}_{12}\mathbf{B}_{22} \\ \mathbf{A}_{21}\mathbf{B}_{11}+\mathbf{A}_{22}\mathbf{B}_{21} \\ \mathbf{A}_{21}\mathbf{B}_{12}+\mathbf{A}_{22}\mathbf{B}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} & \mathbf{A}_{12} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{11} & \mathbf{0} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{0} & \mathbf{A}_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{21} & \mathbf{0} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{B}_{11} \\ \mathbf{B}_{12} \\ \mathbf{B}_{21} \\ \mathbf{B}_{22} \end{bmatrix} = \widetilde{\mathbf{A}} \begin{bmatrix} \mathbf{B}_{11} \\ \mathbf{B}_{12} \\ \mathbf{B}_{21} \\ \mathbf{B}_{22} \end{bmatrix}$$

现假设 $\widetilde{\mathbf{A}}$ 可以分解成 m 个“块秩 1 矩阵”的和：

$$\widetilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} & \mathbf{A}_{12} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{11} & \mathbf{0} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{0} & \mathbf{A}_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{21} & \mathbf{0} & \mathbf{A}_{22} \end{bmatrix} = \sum_{i\in[m]} \begin{bmatrix} \mathbf{P}_{i1} \\ \mathbf{P}_{i2} \\ \mathbf{P}_{i3} \\ \mathbf{P}_{i4} \end{bmatrix} \mathbf{R}_i \begin{bmatrix} \mathbf{Q}_{i1} \\ \mathbf{Q}_{i2} \\ \mathbf{Q}_{i3} \\ \mathbf{Q}_{i4} \end{bmatrix}^\top \tag{1}$$

其中 $\mathbf{R}_i$ 只由 $\mathbf{A}_{11},\mathbf{A}_{12},\mathbf{A}_{21},\mathbf{A}_{22}$ 进行加减运算得到且 $\mathbf{P}_{i1},\ldots,\mathbf{P}_{i4},\mathbf{Q}_{i1},\ldots,\mathbf{Q}_{i4} \in \{\pm\mathbf{I},\mathbf{0}\}$。则

$$\begin{bmatrix} \mathbf{A}_{11}\mathbf{B}_{11}+\mathbf{A}_{12}\mathbf{B}_{21} \\ \mathbf{A}_{11}\mathbf{B}_{12}+\mathbf{A}_{12}\mathbf{B}_{22} \\ \mathbf{A}_{21}\mathbf{B}_{11}+\mathbf{A}_{22}\mathbf{B}_{21} \\ \mathbf{A}_{21}\mathbf{B}_{12}+\mathbf{A}_{22}\mathbf{B}_{22} \end{bmatrix} = \sum_{i\in[m]} \begin{bmatrix} \mathbf{P}_{i1} \\ \mathbf{P}_{i2} \\ \mathbf{P}_{i3} \\ \mathbf{P}_{i4} \end{bmatrix} \mathbf{R}_i \begin{bmatrix} \mathbf{Q}_{i1} \\ \mathbf{Q}_{i2} \\ \mathbf{Q}_{i3} \\ \mathbf{Q}_{i4} \end{bmatrix}^\top \begin{bmatrix} \mathbf{B}_{11} \\ \mathbf{B}_{12} \\ \mathbf{B}_{21} \\ \mathbf{B}_{22} \end{bmatrix} = \sum_{i\in[m]} \begin{bmatrix} \mathbf{P}_{i1} \\ \mathbf{P}_{i2} \\ \mathbf{P}_{i3} \\ \mathbf{P}_{i4} \end{bmatrix} \mathbf{R}_i\mathbf{S}_i = \sum_{i\in[m]} \begin{bmatrix} \mathbf{P}_{i1} \\ \mathbf{P}_{i2} \\ \mathbf{P}_{i3} \\ \mathbf{P}_{i4} \end{bmatrix} \mathbf{T}_i$$

其中 $\mathbf{S}_i = \mathbf{Q}_{i1}\mathbf{B}_{11}+\mathbf{Q}_{i2}\mathbf{B}_{12}+\mathbf{Q}_{i3}\mathbf{B}_{21}+\mathbf{Q}_{i4}\mathbf{B}_{22}$ 只由 $\mathbf{B}_{11},\mathbf{B}_{12},\mathbf{B}_{21},\mathbf{B}_{22}$ 进行加减运算得到。计算全部 m 个 $\mathbf{T}_i = \mathbf{R}_i\mathbf{S}_i$ 会产生 m 个子问题。又 $\mathbf{P}_{i1},\ldots,\mathbf{P}_{i4} \in \{\pm\mathbf{I},\mathbf{0}\}$，因此

$$\begin{bmatrix} \mathbf{A}_{11}\mathbf{B}_{11}+\mathbf{A}_{12}\mathbf{B}_{21} \\ \mathbf{A}_{11}\mathbf{B}_{12}+\mathbf{A}_{12}\mathbf{B}_{22} \\ \mathbf{A}_{21}\mathbf{B}_{11}+\mathbf{A}_{22}\mathbf{B}_{21} \\ \mathbf{A}_{21}\mathbf{B}_{12}+\mathbf{A}_{22}\mathbf{B}_{22} \end{bmatrix} = \sum_{i\in[m]} \begin{bmatrix} \mathbf{P}_{i1} \\ \mathbf{P}_{i2} \\ \mathbf{P}_{i3} \\ \mathbf{P}_{i4} \end{bmatrix} \mathbf{T}_i = \begin{bmatrix} \mathbf{P}_{11}\mathbf{T}_1+\cdots+\mathbf{P}_{m1}\mathbf{T}_m \\ \mathbf{P}_{12}\mathbf{T}_1+\cdots+\mathbf{P}_{m2}\mathbf{T}_m \\ \mathbf{P}_{13}\mathbf{T}_1+\cdots+\mathbf{P}_{m3}\mathbf{T}_m \\ \mathbf{P}_{14}\mathbf{T}_1+\cdots+\mathbf{P}_{m4}\mathbf{T}_m \end{bmatrix}$$

只由 $\mathbf{T}_1,\ldots,\mathbf{T}_m$ 进行加减运算得到。综上，关键就是如何使式 (1) 中的 $m<8$。

下面给出一个 $m=7$ 的分解方法，首先去掉左上的 $\mathbf{A}_{11}$ 和右下的 $\mathbf{A}_{22}$

$$\widetilde{\mathbf{A}} - \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} & \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{11} & \mathbf{0} & \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{22} & \mathbf{0} & \mathbf{A}_{22} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{22} & \mathbf{0} & \mathbf{A}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{A}_{12}-\mathbf{A}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{11}-\mathbf{A}_{22} & \mathbf{0} & \mathbf{A}_{12}-\mathbf{A}_{22} \\ \mathbf{A}_{21}-\mathbf{A}_{11} & \mathbf{0} & \mathbf{A}_{22}-\mathbf{A}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{21}-\mathbf{A}_{22} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{11}-\mathbf{A}_{22} & \mathbf{A}_{11}-\mathbf{A}_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{22}-\mathbf{A}_{11} & \mathbf{A}_{22}-\mathbf{A}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{A}_{12}-\mathbf{A}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{22}-\mathbf{A}_{11} & \mathbf{A}_{12}-\mathbf{A}_{22} \\ \mathbf{A}_{21}-\mathbf{A}_{11} & \mathbf{A}_{11}-\mathbf{A}_{22} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{21}-\mathbf{A}_{22} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$
= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{11}-\mathbf{A}_{22} & \mathbf{A}_{11}-\mathbf{A}_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{22}-\mathbf{A}_{11} & \mathbf{A}_{22}-\mathbf{A}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{22}-\mathbf{A}_{12} & \mathbf{A}_{12}-\mathbf{A}_{22} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{A}_{12}-\mathbf{A}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{12}-\mathbf{A}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}
$$

$$
+ \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{21}-\mathbf{A}_{11} & \mathbf{A}_{11}-\mathbf{A}_{21} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{21}-\mathbf{A}_{22} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{21}-\mathbf{A}_{22} & \mathbf{0} & \mathbf{0} \end{bmatrix}
$$

$$
\Longrightarrow \widetilde{\mathbf{A}} = \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \\ \mathbf{I} \\ \mathbf{0} \end{bmatrix} \mathbf{A}_{11} \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \\ \mathbf{I} \\ \mathbf{0} \end{bmatrix}^\top + \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \\ \mathbf{0} \\ \mathbf{I} \end{bmatrix} \mathbf{A}_{22} \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \\ \mathbf{0} \\ \mathbf{I} \end{bmatrix}^\top + \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \\ -\mathbf{I} \\ \mathbf{0} \end{bmatrix} (\mathbf{A}_{11}-\mathbf{A}_{22}) \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \\ \mathbf{I} \\ \mathbf{0} \end{bmatrix}^\top + \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} (\mathbf{A}_{12}-\mathbf{A}_{22}) \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ -\mathbf{I} \\ \mathbf{I} \end{bmatrix}^\top
$$

$$
+ \begin{bmatrix} \mathbf{I} \\ \mathbf{I} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} (\mathbf{A}_{11}-\mathbf{A}_{12}) \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ -\mathbf{I} \\ \mathbf{0} \end{bmatrix}^\top + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{I} \\ \mathbf{0} \end{bmatrix} (\mathbf{A}_{11}-\mathbf{A}_{21}) \begin{bmatrix} -\mathbf{I} \\ \mathbf{I} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}^\top + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{I} \\ \mathbf{I} \end{bmatrix} (\mathbf{A}_{21}-\mathbf{A}_{22}) \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}^\top
$$

3 算法实现

根据上面的分解易知计算

$$
\begin{bmatrix} \mathbf{S}_1 \\ \mathbf{S}_2 \\ \mathbf{S}_3 \\ \mathbf{S}_4 \\ \mathbf{S}_5 \\ \mathbf{S}_6 \\ \mathbf{S}_7 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{I} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ -\mathbf{I} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{B}_{11} \\ \mathbf{B}_{12} \\ \mathbf{B}_{21} \\ \mathbf{B}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{11}+\mathbf{B}_{21} \\ \mathbf{B}_{12}+\mathbf{B}_{22} \\ \mathbf{B}_{12}+\mathbf{B}_{21} \\ \mathbf{B}_{22}-\mathbf{B}_{21} \\ -\mathbf{B}_{21} \\ \mathbf{B}_{12}-\mathbf{B}_{11} \\ \mathbf{B}_{12} \end{bmatrix}, \quad \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \\ \mathbf{R}_4 \\ \mathbf{R}_5 \\ \mathbf{R}_6 \\ \mathbf{R}_7 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} \\ \mathbf{A}_{22} \\ \mathbf{A}_{11}-\mathbf{A}_{22} \\ \mathbf{A}_{12}-\mathbf{A}_{22} \\ \mathbf{A}_{11}-\mathbf{A}_{12} \\ \mathbf{A}_{11}-\mathbf{A}_{21} \\ \mathbf{A}_{21}-\mathbf{A}_{22} \end{bmatrix}
$$

共会产生 10 次加减运算，计算 $\mathbf{T}_1 = \mathbf{R}_1\mathbf{S}_1, \ldots, \mathbf{T}_7 = \mathbf{R}_7\mathbf{S}_7$ 共会产生 7 个子问题，最后计算

$$
\begin{bmatrix} \mathbf{A}_{11}\mathbf{B}_{11}+\mathbf{A}_{12}\mathbf{B}_{21} \\ \mathbf{A}_{11}\mathbf{B}_{12}+\mathbf{A}_{12}\mathbf{B}_{22} \\ \mathbf{A}_{21}\mathbf{B}_{11}+\mathbf{A}_{22}\mathbf{B}_{21} \\ \mathbf{A}_{21}\mathbf{B}_{12}+\mathbf{A}_{22}\mathbf{B}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \\ \mathbf{I} \\ \mathbf{0} \end{bmatrix} \mathbf{T}_1 + \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \\ \mathbf{0} \\ \mathbf{I} \end{bmatrix} \mathbf{T}_2 + \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \\ -\mathbf{I} \\ \mathbf{0} \end{bmatrix} \mathbf{T}_3 + \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{T}_4 + \begin{bmatrix} \mathbf{I} \\ \mathbf{I} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{T}_5 + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{I} \\ \mathbf{0} \end{bmatrix} \mathbf{T}_6 + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{I} \\ \mathbf{I} \end{bmatrix} \mathbf{T}_7
$$

$$
= \begin{bmatrix} \mathbf{T}_1+\mathbf{T}_5 \\ \mathbf{T}_2+\mathbf{T}_3+\mathbf{T}_4+\mathbf{T}_5 \\ \mathbf{T}_1-\mathbf{T}_3+\mathbf{T}_6+\mathbf{T}_7 \\ \mathbf{T}_2+\mathbf{T}_7 \end{bmatrix}
$$

共会产生 8 次加减运算。

综上，一共会产生 7 个子问题和 18 次加减运算，此时递推关系变成

$$T(n) = 7 \cdot T(n/2) + \frac{18}{4} c_1 n^2$$

设 $n = 2^k$，则

$$\begin{aligned} T(2^k) &= 7^1 \cdot T(2^{k-1}) + \frac{18}{4} c_1 4^k \\ 7^1 \cdot T(2^{k-1}) &= 7^2 \cdot T(2^{k-2}) + 7^1 \frac{18}{4} c_1 4^{k-1} \\ 7^2 \cdot T(2^{k-2}) &= 7^3 \cdot T(2^{k-3}) + 7^2 \frac{18}{4} c_1 4^{k-2} \\ &\vdots \\ 7^{k-1} \cdot T(2^1) &= 7^k \cdot T(2^0) + 7^{k-1} \frac{18}{4} c_1 4^1 \end{aligned}$$

注意 $7^k = (2^{\lg 7})^k = (2^k)^{\lg 7} = n^{\lg 7} \approx n^{2.81}$，累加可得

$$T(n) = c_2 n^{\lg 7} + \frac{18}{4} c_1 4^k \frac{1 - (7/4)^k}{1 - (7/4)} = c_2 n^{\lg 7} + 6c_1 (n^{\lg 7} - n^2) = (c_2 + 6c_1)\, n^{\lg 7} - 6c_1 n^2$$